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On asymptotic approximations to entire functions

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Abstract
A way of circumventing the obstacles in the realization of original ideas by von Neumann and Gabor that are posed by the Balian–Low theorem on localization is shown, by using a special entire function with a strong exponential localization property. A square-integrable, doubly periodic and exponentially localized basis in Hilbert space of functions on **C** is used to solve the problem of asymptotic approximations to entire functions, in Hilbert space metrics. A new technique is suggested for numerical methods in phase-space quantum mechanics and signal processing.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

It is a well-known fact that entire function $f(z)$ is uniquely determined by its values $f(z_n)$ on an infinite convergent set $\{z_n\}$, and is not determined by its values $f(z_{m,n})$ on, say, a rectangular lattice $\{z_{m,n}\} = \{am + \mathrm{i}bn\}$, whatever its density $(ab)^{-1}$. When restricted to Hilbert space $E_2(\mathbb{C})$ of entire functions with norm

$$\|f\|^2 = \iint \mathrm{e}^{-y^2} |f(x+\mathrm{i}y)|^2 \,\mathrm{d}x\,\mathrm{d}y, \tag{1}$$

instead of function values, $f(z_{m,n})$, one may deal with the scalar products

$$f_{m,n} = \iint \mathrm{e}^{-y^2}\, \overline{a_{m,n}(x+\mathrm{i}y)}\, f(x+\mathrm{i}y)\,\mathrm{d}x\,\mathrm{d}y, \tag{2}$$

with $a_{m,n}(z)$ representing the shifts, $a(z+am+\mathrm{i}bn)$, of a single entire function combined with simple exponential multipliers. (Note that, say, in signal processing, these two approaches are, in practice, equivalent: the 'amplitudes' obtained in the course of sampling *are* scalar products, and the set of $f_{m,n}$ may be viewed as a discretized integral transform, closely related to a Gabor transform or a wavelet transform.) Another well-known fact consists of the separability of the functional space under consideration, that is, in the existence of countable bases. When the set (2) is associated with the commutative discrete subgroup of a Weyl–Heisenberg group (WHG) representation in the space of entire functions, the density has a critical value, $(ab)^{-1} = (2\pi)^{-1}$, so that we have a basis, and approximations to entire functions may be constructed in Hilbert space metrics. (The respective *density of states* plays an important role in quantum statistical mechanics.) The first analysis along these lines goes back to von Neumann [1], and here the close connection to the theory of theta functions should be noted (see, for instance, [2]). The idea of elementary signals put forward by Gabor is another formulation of the same problem. Further developments include the construction of the kq representation by Zak[3]. However, in 1981, Balian [4] proved that the bases associated with the von Neumann lattice are singular when a mutual orthogonality additional condition is imposed, that is, the product of respective moments is infinite: $\Delta x \Delta y = \infty$. A similar fact in connection with the quantum Hall effect was established by Low [5]. The result is known as the Balian–Low theorem and plays a central role in modern harmonic analysis (see, for instance, [6]). Thus, the countable localized bases in a form of a lattice in **C**, which could be used for localized expansions, are forbidden. In an attempt to overcome this difficulty, the author has constructed a basis using an entire function $a(z)$, which is strongly exponentially localized, and serves as a basis for the representation of the discrete *anticommutative* subgroup of the WHG[7]. The set is twice overcomplete, and has an overall density $(ab)^{-1} = (\pi)^{-1}$; it consists of a quartet of slightly nonorthogonal sublattices with a density of $(4\pi)^{-1}$ each. Asymptotic expansions for entire functions using the quartet of sublattices give a solution to the problem of localized approximations, in a manner ready for practical applications. The procedure uses uniform *discretized* expansions, in contrast with other approaches, like coherent-state continual expansions.

As examples of other possibilities of circumventing the Balian–Low theorem, one may mention the existence of localized bases of wavelets [8], and a procedure of constructing an orthonormal basis for the lowest Landau level [9, 10].

The asymptotic approximation problem formulation may be refined as follows. Given that the entire function is represented by the scalar products (2) on a lattice on a finite number of sites in a rectangular domain, $2aM \times 2bN$, and that their values are exponentially small close to the domain boundaries, then it can be reconstructed so that this *local* reconstruction is exponentially small in the rest of the complex plane, in the sense of Hilbert space norm (1).

2. Asymptotic approximation algorithm using quantum phase-space fabric

The great advantage of performing calculations with theta functions is in their very fast convergence rate; the same is true with the algorithm described in the present section, another advantage being its parallelism.

2.1. The basis and the pseudospin degree of freedom

The expressions for WHG action take a more compact form when (single variable) entire functions $f(z)$ that belong to $E_2(\mathbb{C})$ are represented as functions of two real variables:

$$f(z) \to f(x, y) = \mathrm{e}^{-\frac{1}{2}y^2} f(x + \mathrm{i}y). \tag{3}$$

Such representation for entire functions is convenient for quantum mechanics in the lowest Landau level, and the WHG represents the symmetry of the problem with respect to magnetic translations. (By using single notation for the functions in (3) and throughout the present paper, the fact that these are viewed as different representations of a single element of a separable Hilbert space, up to a unitary transformation is emphasized.)

The generalized basis in $E_2(\mathbb{C})$ introduced in [7] is represented by a pair of function sets:

$$\psi_{\mu,k_x,k_y}(x,y) = \mathrm{e}^{-k_x{}^2/2-y^2/2+k_x(y-\mathrm{i}x)}\,\vartheta_\mu{}^2\left(\frac{\sqrt{\pi}}{2}(k_y - \mathrm{i}k_x + x + \mathrm{i}y)\right), \tag{4}$$

either with a pair $\mu = 3, 1$, or a pair $\mu = 4, 2$. The functions (4) are related closely to those constructed by Zak [3] for $L_2(\mathbb{R})$. Another important set of Bloch functions, which form a quartet,

$$\varphi_{\mu,k_x,k_y}(x,y) = \mathrm{e}^{-x^2/4-y^2/4-\mathrm{i}xy/2}\,\vartheta_{\overline{\mu}}\left(\frac{\sqrt{\pi}}{2}(2k_x + \mathrm{i}x - y)\right)\vartheta_\mu\left(\frac{\sqrt{\pi}}{2}(2k_y + x + \mathrm{i}y)\right), \tag{5}$$

serves for the projection of the functions, as elements of Hilbert space, locally to the given sublattice of the quartet (the bar in $\overline{\mu}$ denotes conjugation with respect to Jacobi imaginary transformation). The set (5), in contrast with (4), is nonorthogonal.

Interpretation of the interrelation between the two sets (4) and (5) is performed using pseudospin degree of freedom, and it leads to an exponentially localized set of magnetic Wannier functions using orthogonalization procedure with the help of the normalization function

$$N(k_x,k_y) \propto (\vartheta_3(\sqrt{\pi}k_x)\vartheta_3(\sqrt{\pi}k_y))^{1/2}. \tag{6}$$

For more details see paper [7]. The scalar products (2) are constructed using a magnetic Wannier function $a(z)$. Since this construction plays a central role in the present discussion, some remarks concerning calculational details for the function $a(z)$ are presented in the following subsection.

2.2. *The function* $a(z)$

With the help of an auxiliary periodic entire function $\alpha(z)$,

$$\alpha(z) = 1 + 2\sum_{n=1}^{\infty}(-1)^n\alpha_n\,\mathrm{e}^{-\pi n}\,\mathrm{e}^{-\pi n^2}\cos(\sqrt{\pi}nz), \tag{7}$$

$a(z)$ is written in the form

$$a(z) = \mathrm{e}^{-\frac{1}{4}z^2}\,\alpha(z)\alpha(\mathrm{i}z). \tag{8}$$

The periodic entire function $\alpha(z)$ (7) has the following integral representation:

$$\alpha(z) \propto \int \frac{\vartheta_4(x + z\sqrt{\pi}/2)}{\sqrt{\vartheta_4(x)}}\,\mathrm{d}x; \tag{9}$$

the integral is taken over an interval of length π on the real axis. We restrict ourselves to the lemniscate case of elliptic functions, and by $\vartheta_\mu(x)$ we mean Jacobi theta function $\vartheta_\mu(x, \mathrm{e}^{-\pi})$. (Function (9) may be generalized to its p-analog useful for Sobolev functional spaces:

$$\alpha_p(z) \propto \int \vartheta_4(x + z\sqrt{\pi}/2)(\vartheta_4(x))^{-\frac{1}{p}}\,\mathrm{d}x. \tag{10}$$

An explicit expression for Fourier series coefficients for $p = 1$ may be found in [12].)

The coefficients α_n are defined using the following Fourier expansion:

$$\frac{1}{\sqrt{\vartheta_4(z)}} \propto 1 + 2\sum_{n=1}^{\infty}\alpha_n\,\mathrm{e}^{-\pi n}\cos(2nz), \tag{11}$$

and are calculated numerically; the factor $\mathrm{e}^{-\pi n}$ in the sum above is responsible for strong exponential localization; the first ten values for α_n are {0.501 052, 0.375 586, 0.312 961, 0.273 833, 0.246 446, 0.225 907, 0.209 77, 0.196 654, 0.185 718, 0.174 832}. By substituting Fourier series for the theta function into (9) we arrive at (7).

Table 1. Four copies of a function are processed in parallel and then their Fourier components are correlated.

Function copies	Local amplitudes	Fourier transforms	Projection: $a_\mu f_\mu = 0$ and $b_\mu f_\mu = 0$	
			$a_\mu(k_x, k_y)$	$b_\mu(k_x, k_y)$
f	$f_{2m,2n}$	$f_3(k_x, k_y)$	$-\psi_{3,k_x,k_y}(0,0)$	$\psi_{1,k_x,k_y}(0,0)$
f	$f_{2m-1,2n-1}$	$f_1(k_x, k_y)$	$\psi_{1,k_x,k_y}(0,0)$	$\psi_{3,k_x,k_y}(0,0)$
f	$f_{2m-1,2n}$	$f_4(k_x, k_y)$	$\psi_{4,k_x,k_y}(0,0)$	$-\psi_{2,k_x,k_y}(0,0)$
f	$f_{2m,2n-1}$	$f_2(k_x, k_y)$	$\psi_{2,k_x,k_y}(0,0)$	$\psi_{4,k_x,k_y}(0,0)$

2.3. Reconstruction algorithm

Given an arbitrary entire function $f(z)$, square integrable in the sense of (1), approximations to the four sets of 'amplitudes' $f_{m,n}$ may be calculated with the help of (2), in a *finite* domain, and this key feature is due to strong exponential localization. These 'amplitudes' do represent the function with some precision, and further processing relates them to the mutual nonorthogonality of different sublattices mentioned in section 2.1.

Reconstruction includes the following steps:

- (i) As a first step, calculation of the 'amplitudes' $f_{m,n}$, and dividing them into four sets related to the quartet (5).
- (ii) As a next step, Fourier transformation is performed, separately in each set of the quartet, to obtain four periodic functions, which are coupled by two linear relations, and do represent projections to (5).
- (iii) To satisfy the latter relations, pointwise projection of the four functions onto two complementary subspaces, which do respect both the refined functional values and the error to approximation.
- (iv) After refining the four Fourier transforms, smoothing procedure in (k_x, k_y) for every component may be applied. This possibility is a consequence of the general fact that smooth Fourier transforms do represent localized functions.

Reconstruction, which is summarized in table 1, includes, therefore, error correction, due to double overcompleteness. Special attention must be paid to the relative phases of the four functions, for instance, using conventions of [7]. The quartet of refined periodic functions $\{f_\mu(k_x, k_y)\}$ with a fundamental domain of area π during reconstruction is thus transformed into a pair of quasiperiodic functions defined in a fundamental domain of area 4π, with the help of the normalization function (6).

The projection matrix, acting on the four functions $\{f_\mu(k_x, k_y)\}$, $\mu = 1, 2, 3, 4$, which represents the erroneous part, reads as:

$$\frac{1}{|w|^2+1}\begin{pmatrix} \frac{1}{2}(w\bar{w}+1) & \frac{-\bar{w}w+w+\bar{w}+1}{2\sqrt{2}} & \frac{1}{2}(\bar{w}-w) & \frac{\bar{w}w+w+\bar{w}-1}{2\sqrt{2}} \\ \frac{-\bar{w}w+w+\bar{w}+1}{2\sqrt{2}} & \frac{1}{2}(w\bar{w}+1) & \frac{\bar{w}w+w+\bar{w}-1}{2\sqrt{2}} & \frac{1}{2}(\bar{w}-w) \\ \frac{1}{2}(\bar{w}-w) & \frac{\bar{w}w+w+\bar{w}-1}{2\sqrt{2}} & \frac{1}{2}(w\bar{w}+1) & \frac{\bar{w}w-w-\bar{w}-1}{2\sqrt{2}} \\ \frac{\bar{w}w+w+\bar{w}-1}{2\sqrt{2}} & \frac{1}{2}(\bar{w}-w) & \frac{\bar{w}w-w-\bar{w}-1}{2\sqrt{2}} & \frac{1}{2}(w\bar{w}+1) \end{pmatrix}, \tag{12}$$

where w is defined using Jacobi elliptic function sd by

$$w(k_x + \mathrm{i}k_y) = \frac{1}{2} sd^2(k_x + \mathrm{i}k_y|1/2) = \frac{\psi_{1,k_x,k_y}(0,0)}{\psi_{3,k_x,k_y}(0,0)}. \tag{13}$$

The complementary projection matrix gives refined Fourier components and this final stage is to represent the asymptotic approximation using generalized basis (4).

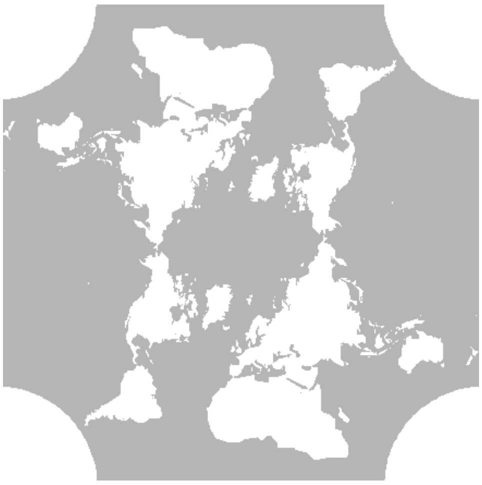

Figure 1. World map on a torus. This is a double cover map with four singular points, and it is a tessellation.

When used for signal processing, the method described above is expected to give substantial reduction of noise. Within any of the four complex channels, the signal may be processed in parallel, since the basis functions within each of the four sublattices are orthonormal.

2.4. The map

The reconstruction algorithm described above relies upon the connection between translations in the complex plane and three-dimensional rotations, using pseudospin degree of freedom introduced in [7]. The map (13) is visualized, in this connection, with the help of stereographic projection,

$$r(w) = \left\{ \frac{2\Re(w)}{|w|^2+1}, \frac{2\Im(w)}{|w|^2+1}, \frac{|w|^2-1}{|w|^2+1} \right\}. \tag{14}$$

The respective map from torus to sphere is a Hopf map from two-dimensional translation-invariant generalized subspace of Hilbert space to the complex plane (13) composed with stereographic projection. It is illustrated with the help of a double-cover world map in figure 1. The map is a tessellation of the plane. There are four singular points, two of them being poles, and the other two—equatorial and the equator—not shown, has a shape of a square.

The importance of symmetric spaces is well known; by mapping an entire function to a torus, the accuracy of the resulting approximation becomes independent of the particular position of the lattice chosen.

3. Bargmann representation for $L_2(\mathbb{R})$ elements

The purpose of the present section is to emphasize the fact that the results of the present paper are by no means restricted to the functional space $E_2(\mathbb{C})$ with norm (1). The one-to-one correspondence between generalized bases parameterized by p,

$$\mathrm{e}^{-\mathrm{i}px} \leftrightarrow \mathrm{e}^{-\frac{1}{2}p^2-\frac{1}{2}y^2-\mathrm{i}pz}, \tag{15}$$

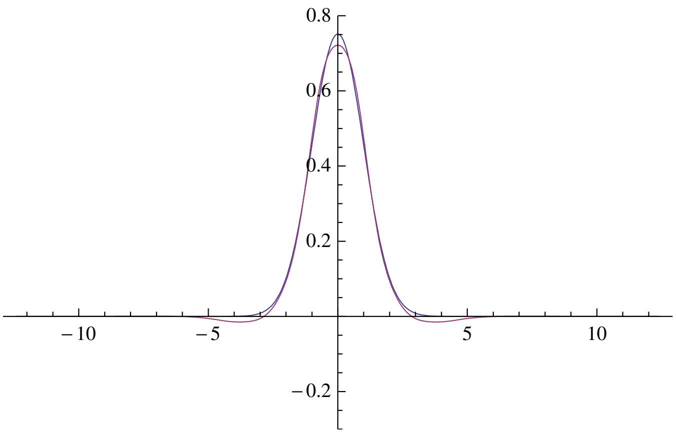

Figure 2. The function $a(p)$ as compared to coherent state wavefunction; the functions are normalized to 1.

defines effectively Bargmann transformation, a unitary mapping between $L_2(\mathbb{R})$ and $E_2(\mathbb{C})$. Our form (15) is slightly different from the standard one, as in [11], and is chosen to have translations acting in the same way both in $L_2(\mathbb{R})$, and $E_2(\mathbb{C})$, as shifts by a: T_a: $f(x) \to f(a+x)$ and $f(z) \to f(a+z)$. Recall that Fourier transformation is unitary on the former space, and that it maps translations to modulations and vice versa. (These take the form of 'magnetic translations' in the complex plane, closely related to the modular group, in the theory of Jacobi theta functions (see [2]; an extremely useful source of practically important relations is [12])). So, the correspondence (15) is compatible with WHG action that is very important in applications, in particular, in phase-space quantum mechanics and signal processing.

The function $a(z)$ from $E_2(\mathbb{C})$ is transformed into $L_2(\mathbb{R})$ function $a(p)$,

$$a(p) = \frac{a_0\,\mathrm{e}^{-\frac{p^2}{2}}}{\sqrt{\vartheta_3(p\sqrt{\pi}, \mathrm{e}^{-\pi})}} \left(1 + 2\sum_{n=1}^{\infty}(-1)^n\,\mathrm{e}^{-2\pi n^2 - \pi n}\cosh\left(2np\sqrt{\pi}\right)\alpha_n\right), \tag{16}$$

with normalization constant $a_0 = 0.752\,182$. The function (16) coincides with its own Fourier transform. The coefficients α_n are defined as Fourier coefficients of the Fourier expansion of the inverse square root of Jacobi theta function by (11). Their comparison with coherent state wavefunctions is presented by figures 2 and 3. A more symmetric representation, using Jacobi imaginary transformation, reads as follows:

$$a(p) = a_0 \frac{1 + 2\sum_{n=1}^{\infty}(-1)^n\,\mathrm{e}^{-2\pi n^2 - \pi n}\cosh(2np\sqrt{\pi})\alpha_n}{\sqrt{1 + 2\sum_{n=1}^{\infty}\mathrm{e}^{-\pi n^2}\cosh(2np\sqrt{\pi})}}. \tag{17}$$

Expression (17) may be used for Padé style approximations in $\cosh(p\sqrt{\pi})$.

Invariance with respect to Fourier transformation explains the restriction in the present discussion to the so-called equiharmonic or lemniscate case of elliptic functions; many facts related to this case were known to Gauss, and, in particular, in its connection to Bernoulli lemniscate geometry.

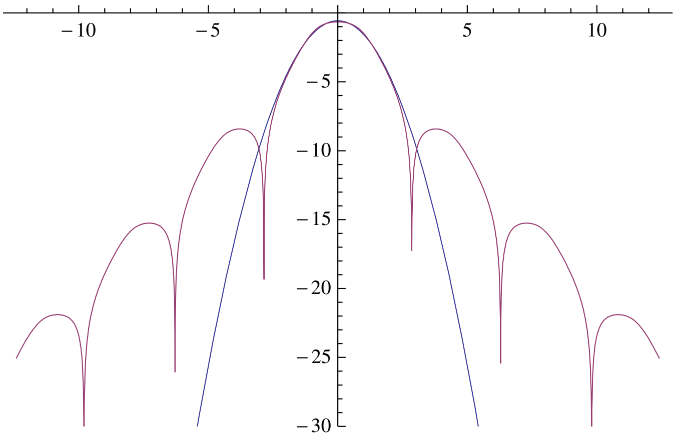

Figure 3. Natural logarithms of the respective probability distributions compared.

4. Concluding remarks

Many applications of the method described in the present paper seem to be interesting. Two of them are briefly described below.

4.1. The problem of recognition of the standard function in a shifted position in the complex plane

In many signal processing applications the recognition of a standard signal shifted in the complex plane, i.e. shifted and 'modulated', is extremely useful. Calling it $a(x, y)$, the problem reduces to calculations with single function $A_{m,n}(\xi, \eta)$ that is represented by an integral

$$\iint \mathrm{e}^{-y^2}\, \overline{a_{m,n}(x, y)}\, \mathrm{e}^{\mathrm{i}\eta x}\, a(x + \xi, y + \eta)\, \mathrm{d}x\, \mathrm{d}y. \tag{18}$$

It is clear that the restrictions of the Heisenberg uncertainty relation are completely overcome, in this problem, with the reconstruction algorithm from section 2.3. The calculations using the inverse Bargmann transformation seem to be good as well.

4.2. von Neumann lattice and Gabor frame

Hamiltonian phase-space formulation of the classical mechanics is supported by Liouville's theorem on volume conservation, and plays an important role in the formulation of the statistical mechanics: probabilistic sample space for a single mechanical degree of freedom forms a plane, and distribution functions are normalized:

$$\iint f(x, p)\, \mathrm{d}x\, \mathrm{d}p = 1. \tag{19}$$

When going to quantum mechanics, which includes probabilistic description and quantization, one could expect that the integral (19) would take the form of a sum:

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f_{m,n} = 1. \tag{20}$$

That is the original idea of von Neumann, and its analog is Gabor's idea of 'elementary' signals. The states labeled by m, n in (20) have to represent orthonormal basis in order to form a quantum sample space. The restrictions due to the Balian–Low theorem make such traces over the von Neumann–Gabor lattice divergent. It may be easily verified that a sum taken with the help of the double overcomplete set used in the present paper,

$$\frac{1}{2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}|a_{m,n}\rangle\langle a_{m,n}|, \tag{21}$$

is a true resolution of identity with strong localization property. Its proof is quite simple with the help of Pauli matrices in generalized invariant subspaces of the representation of the anticommutative subgroup of the WHG, that was constructed in [7]. This gives a solution to the problem of the quantum sample space.

The resolution of identity (21) is equally applicable to the time–frequency plane. Gabor's elementary signals may be understood now in the framework of this double overcomplete set.

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